

SEMIAMPLE INVERTIBLE SHEAVES WITH SEMIPOSITIVE CONTINUOUS HERMITIAN METRICS

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ABSTRACT. Let (L, h) be a pair of a semiample invertible sheaf and a semipositive continuous hermitian metric on a proper algebraic variety. In this paper, we prove that (L, h) is semiample metrized, which is a generalization of the question due to S. Zhang.

INTRODUCTION

Let X be a proper algebraic variety over \mathbb{C} . Let L be an invertible sheaf on X and let h be a continuous hermitian metric of L . We say that (L, h) is *semiample metrized* if, for any $\epsilon > 0$, there is $n > 0$ such that, for any $x \in X(\mathbb{C})$, we can find $l \in H^0(X, L^{\otimes n}) \setminus \{0\}$ with

$$\sup \{h^{\otimes n}(l, l)(w) \mid w \in X(\mathbb{C})\} \leq e^{\epsilon n} h^{\otimes n}(l, l)(x).$$

In the paper [5], Shouwu Zhang proposed the following question:

Question 0.1. [5, Question 3.6] If L is ample and h is smooth and semipositive, then does it follow that (L, h) is semiample metrized?

In [5, Theorem 3.5], he actually gave the affirmative answer in the case where X is smooth over \mathbb{C} . The purpose of this paper is to give an answer for a generalization of the above question. First of all, we fix notations: We say that L is *semiample* if there is a positive integer n_0 such that $L^{\otimes n_0}$ is generated by global sections. Moreover, h is said to be *semipositive* (or we say that (L, h) is semi-positive) if, for any point $x \in X(\mathbb{C})$ and a local basis s of L on a neighborhood of x , $-\log h(s, s)$ is plurisubharmonic around x (for the definition of plurisubharmonicity on a singular variety, see Section 1). Note that h is not necessarily smooth. By using the recent work [1] due to Coman, Guedj and Zeriahi, we have the following answer:

Theorem 0.2. *If L is semiample and h is continuous and semipositive, then (L, h) is semiample metrized.*

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1. PLURISUBHARMONIC FUNCTIONS ON SINGULAR COMPLEX ANALYTIC SPACES

Let T be a reduced complex analytic space. An upper-semicontinuous function

$$\varphi : T \rightarrow \mathbb{R} \cup \{-\infty\}$$

is said to be *plurisubharmonic* if $\varphi \not\equiv -\infty$ and, for each $x \in T$, there is an open neighborhood U_x of x together with an open set W_x of \mathbb{C}^{n_x} and a plurisubharmonic function Φ_x on W_x such that U_x is a closed complex analytic subspace of W_x and $\varphi|_{U_x} = \Phi_x|_{U_x}$. For an analytic map $f : T' \rightarrow T$ of reduced complex analytic spaces and a plurisubharmonic function φ on T , it is easy to see that $\varphi \circ f$ is either identically $-\infty$ or plurisubharmonic on T' . By the theorem due to Fornaess and Narasimhan [3, Theorem 5.3.1], an upper-semicontinuous function $\varphi : T \rightarrow \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic if and only if, for any analytic map $\varrho : \mathbb{D} \rightarrow T$, $\varphi \circ \varrho$ is either identically $-\infty$ or subharmonic on \mathbb{D} , where $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. Moreover, if T is compact and φ is plurisubharmonic on T , then φ is locally constant.

Let ω be a smooth $(1, 1)$ -form on T such that, for each $x \in T$, ω is locally given by $dd^c(u)$ for some smooth function u on a neighborhood of x . Let ϕ be a *quasi-plurisubharmonic function* on T , that is, for each $x \in T$, ϕ can be locally written by the sum of a smooth function and a plurisubharmonic function around x . We say that ϕ is ω -*plurisubharmonic* if there is an open covering $T = \bigcup_\lambda U_\lambda$ together with a smooth function u_λ on U_λ for each λ such that $\omega|_{U_\lambda} = dd^c(u_\lambda)$ and $\phi|_{U_\lambda} + u_\lambda$ is plurisubharmonic on U_λ . The condition for ω -plurisubharmonicity is often denoted by $dd^c([\phi]) + \omega \geq 0$.

Here we consider the following lemma.

Lemma 1.1. *Let $f : X \rightarrow Y$ be a surjective and proper morphism of algebraic varieties over \mathbb{C} . Let φ be a real-valued function on $Y(\mathbb{C})$. Then we have the following:*

- (1) *φ is continuous if and only if $\varphi \circ f$ is continuous.*
- (2) *We assume that φ is continuous. Then φ is plurisubharmonic if and only if $\varphi \circ f$ is plurisubharmonic.*

Proof. (1) It is sufficient to see that if $\varphi \circ f$ is continuous, then φ is continuous. Otherwise, there are $y \in Y(\mathbb{C})$, $\epsilon_0 > 0$ and a sequence $\{y_n\}$ on $Y(\mathbb{C})$ such that $\lim_{n \rightarrow \infty} y_n = y$ and $|\varphi(y_n) - \varphi(y)| \geq \epsilon_0$ for all n . We choose $x_n \in X(\mathbb{C})$ such that $f(x_n) = y_n$. As $f : X \rightarrow Y$ is proper, we can find a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x := \lim_{i \rightarrow \infty} x_{n_i}$ exists in $X(\mathbb{C})$. Note that

$$f(x) = \lim_{i \rightarrow \infty} f(x_{n_i}) = \lim_{i \rightarrow \infty} y_{n_i} = y,$$

so that, as $\varphi \circ f$ is continuous,

$$\varphi(y) = (\varphi \circ f)(x) = \lim_{i \rightarrow \infty} (\varphi \circ f)(x_{n_i}) = \lim_{i \rightarrow \infty} \varphi(f(x_{n_i})) = \lim_{i \rightarrow \infty} \varphi(y_{n_i}),$$

which is a contradiction, so that φ is continuous.

(2) We need to check that if $\varphi \circ f$ is plurisubharmonic, then φ is plurisubharmonic. By using Chow's lemma, we may assume that $f : X \rightarrow Y$ is projective. Moreover, since the assertion is local with respect to Y , we may further assume

that there is a closed embedding $\iota : X \hookrightarrow Y \times \mathbb{P}^N$ such that $p \circ \iota = f$, where $p : Y \times \mathbb{P}^N \rightarrow Y$ is the projection to the first factor. The remaining proof is same as the last part of the proof of [2, Theorem 1.7]. Let $g : (\mathbb{D}, 0) \rightarrow (Y, y)$ be a germ of analytic map. By the theorem due to Fornaess and Narasimhan, it is sufficient to show that $\varphi \circ g$ is subharmonic. Clearly we may assume that g is given by the normalization of a 1-dimensional irreducible germ (C, y) in (Y, y) . Using hyperplanes in \mathbb{P}^N , we can find $x \in X$ and a 1-dimensional irreducible germ (C', x) in (X, x) such that (C', x) is lying over (C, y) . Let $g' : (\mathbb{D}, 0) \rightarrow (X, x)$ be the germ of analytic map given by the normalization of (C', x) . Then we have the analytic map $\sigma : (\mathbb{D}, 0) \rightarrow (\mathbb{D}, 0)$ with $g \circ \sigma = f \circ g'$:

$$\begin{array}{ccc} (\mathbb{D}, 0) & \xrightarrow{g'} & (X, x) \\ \sigma \downarrow & & \downarrow f \\ (\mathbb{D}, 0) & \xrightarrow{g} & (Y, y) \end{array}$$

Changing a variable of $(\mathbb{D}, 0)$, we may assume that σ is given by $\sigma(z) = z^m$ for some positive integer m . Then $\varphi \circ g \circ \sigma$ is subharmonic because $\varphi \circ f$ is plurisubharmonic. Therefore, as σ is étale over the outside of 0, $\varphi \circ g$ is subharmonic on the outside of 0, and hence $\varphi \circ g$ is subharmonic on $(\mathbb{D}, 0)$ by removable singularities of subharmonic functions. \square

2. DESCENT OF SEMIPOSITIVE CONTINUOUS HERMITIAN METRIC

In this section, we consider a descent problem of a semipositive continuous hermitian metric.

Theorem 2.1. *Let $f : X \rightarrow Y$ be a surjective and proper morphism of algebraic varieties over \mathbb{C} with $f_* \mathcal{O}_X = \mathcal{O}_Y$. Let L be an invertible sheaf on Y . If h' is a semipositive continuous hermitian metric of $f^*(L)$, then there is a semipositive continuous hermitian metric h of L such that $h' = f^*(h)$.*

Proof. Let h_0 be a continuous hermitian metric of L on Y . There is a continuous function ϕ on $X(\mathbb{C})$ such that $h' = \exp(\phi) f^*(h_0)$. Let F be a subvariety of X such that F is an irreducible component of a fiber of $f : X \rightarrow Y$. Then, as

$$(f^*(L), h')|_F \simeq (\mathcal{O}_F, \exp(\phi|_F)),$$

we can see that $-\phi|_F$ is plurisubharmonic, so that $\phi|_F$ is constant. Therefore, for any point $y \in Y(\mathbb{C})$, $\phi|_{\mu^{-1}(y)}$ is constant because $\mu^{-1}(y)$ is connected, and hence there is a function ψ on $Y(\mathbb{C})$ such that $\psi \circ f = \phi$. By (1) in Lemma 1.1, ψ is continuous, so that, if we set $h := \exp(\psi) h_0$, then h is continuous on $Y(\mathbb{C})$ and $h' = f^*(h)$.

Finally let us see that h is semipositive. As it is a local question on Y , we may assume that there is a local basis s of L over Y . If we set $\varphi = -\log h(s, s)$, then $\varphi \circ f$ is plurisubharmonic because h' is semipositive. Therefore, by (2) in Lemma 1.1, φ is plurisubharmonic, as required \square

3. THE PROOF OF THEOREM 0.2

In the case where X is smooth over \mathbb{C} , L is ample and h is smooth, this theorem was proved by Zhang [5, Theorem 3.5]. First we assume that L is ample. Then there are a positive integer n_0 and a closed embedding $X \hookrightarrow \mathbb{P}^N$ such that $\mathcal{O}_{\mathbb{P}^N}(1)|_X \simeq L^{\otimes n_0}$. Let h_{FS} be the Fubini-Study metric of $\mathcal{O}_{\mathbb{P}^N}(1)$. Let ϕ be the continuous function on $X(\mathbb{C})$ given by $h^{\otimes n_0} = \exp(-\phi) h_{FS}|_X$. We set $\omega = c_1(\mathcal{O}_{\mathbb{P}^N}(1), h_{FS})$. Then ϕ is $(\omega|_X)$ -plurisubharmonic. Therefore, by [1, Corollary C], there is a sequence $\{\varphi_i\}$ of smooth functions on $\mathbb{P}^N(\mathbb{C})$ with the following properties:

- (1) φ_i is ω -plurisubharmonic for all i .
- (2) $\varphi_i \geq \varphi_{i+1}$ for all i .
- (3) For $x \in X(\mathbb{C})$, $\lim_{i \rightarrow \infty} \varphi_i(x) = \phi(x)$.

Since X is compact and ϕ is continuous, (3) implies that the sequence $\{\varphi_i\}$ converges ϕ uniformly on $X(\mathbb{C})$. We choose i such that $|\phi(x) - \varphi_i(x)| \leq \epsilon n_0/2$ for all $x \in X$. We set $h_i = \exp(-\varphi_i) h_{FS}$. Then h_i is a semipositive smooth hermitian metric of $\mathcal{O}_{\mathbb{P}^N}(1)$. Therefore, there is a positive integer n_1 such that, for $x \in \mathbb{P}^N(\mathbb{C})$, we can find $l \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n_1)) \setminus \{0\}$ with

$$\sup \left\{ h_i^{\otimes n_1}(l, l)(w) \mid w \in \mathbb{P}^N(\mathbb{C}) \right\} \leq e^{n_1(\epsilon n_0/2)} h_i^{\otimes n_1}(l, l)(x).$$

In particular, if $x \in X(\mathbb{C})$, then $l(x) \neq 0$ (so that $l|_X \neq 0$) and

$$\sup \left\{ h_i^{\otimes n_1}(l, l)(w) \mid w \in X(\mathbb{C}) \right\} \leq e^{\epsilon n_0 n_1/2} h_i^{\otimes n_1}(l, l)(x).$$

Note that

$$(3.1) \quad h^{\otimes n_0} e^{-\epsilon n_0/2} \leq h_i \leq h^{\otimes n_0}$$

on $X(\mathbb{C})$ because $h_i = h^{\otimes n_0} \exp(\phi - \varphi_i)$ and $-\epsilon n_0/2 \leq \phi - \varphi_i \leq 0$ on $X(\mathbb{C})$. Therefore,

$$\sup \left\{ h^{\otimes n_0 n_1}(l, l)(w) \mid w \in X(\mathbb{C}) \right\} e^{-n_0 n_1 \epsilon/2} \leq \sup \left\{ h_i^{\otimes n_1}(l, l)(w) \mid w \in X(\mathbb{C}) \right\}$$

and

$$h_i^{\otimes n_1}(l, l)(x) \leq h^{\otimes n_0 n_1}(l, l)(x),$$

and hence

$$\sup \left\{ h^{\otimes n_0 n_1}(l, l)(w) \mid w \in X(\mathbb{C}) \right\} \leq e^{n_1 n_0 \epsilon} h^{\otimes n_0 n_1}(l, l)(x),$$

as required.

In general, as L is semiample, there are a positive integer n_2 , a projective algebraic variety Y over \mathbb{C} , a morphism $f : X \rightarrow Y$ and an ample invertible sheaf A on Y such that $f_* \mathcal{O}_X = \mathcal{O}_Y$ and $f^*(A) \simeq L^{\otimes n_2}$. Thus, by Theorem 2.1, there is a semipositive continuous hermitian metric k of A such that $(f^*(A), f^*(k)) \simeq (L^{\otimes n_2}, h^{\otimes n_2})$. Therefore, the assertion of the theorem follows from the previous observation.

4. A VARIANT OF THEOREM 0.2

The following theorem is a consequence of Theorem 0.2 together with the arguments in [5, Theorem 3.3]. However, we can give a direct proof using ideas in the proof of Theorem 0.2.

Theorem 4.1. *Let X be a projective algebraic variety over \mathbb{C} . Let L be an ample invertible sheaf on X and let h be a semipositive continuous hermitian metric of L . Let us fix a reduced subscheme Y of X , $l \in H^0(Y, L|_Y)$ and a positive number ϵ . Then, for the given X, L, h, Y, l and ϵ , there is a positive integer n_1 such that, for all $n \geq n_1$, we can find $l' \in H^0(X, L^{\otimes n})$ with $l'|_Y = l^{\otimes n}$ and*

$$\sup \{h^{\otimes n}(l', l')(w) \mid w \in X(\mathbb{C})\} \leq e^{n\epsilon} \sup \{h(l, l)(w) \mid w \in Y(\mathbb{C})\}^n.$$

Proof. In the case where X is smooth over \mathbb{C} and h is smooth and positive, the assertion of the theorem follows from [5, Theorem 2.2], in which Y is actually assumed to be a subvariety of X . However, the proof works well under the assumption that Y is a reduced subscheme. First of all, let us see the theorem in the case where X is smooth over \mathbb{C} and h is smooth and semipositive. As L is ample, there is a positive smooth hermitian metric t of L with $t \leq h$. Let us choose a positive integer m such that $e^{-\epsilon/2} \leq (t/h)^{1/m} \leq 1$ on $X(\mathbb{C})$. If we set $t_m = h^{1-1/m} t^{1/m}$, then t_m is smooth and positive, so that, for a sufficiently large integer n , there is $l' \in H^0(X, L^{\otimes n})$ such that $l'|_Y = l^{\otimes n}$ and

$$\sup \{t_m^{\otimes n}(l', l')(w) \mid w \in X(\mathbb{C})\} \leq e^{n\epsilon/2} \sup \{t_m(l, l)(w) \mid w \in Y(\mathbb{C})\}^n,$$

and hence the assertion follows because $e^{-\epsilon/2}h \leq t_m \leq h$ on $X(\mathbb{C})$.

For a general case, we use the same symbols $n_0, X \hookrightarrow \mathbb{P}^N, h_{FS}, \phi, \omega$ and $\{\varphi_i\}$ as in the proof of Theorem 0.2. Clearly we may assume that $l \neq 0$. Since L is ample, if a_0 is sufficiently large integer, then, for each $j = 0, \dots, n_0 - 1$, there is $l_j \in H^0(X, L^{\otimes n_0 a_0 + j})$ with $l_j|_Y = l^{\otimes n_0 a_0 + j}$. Let us fix a positive number A such that

$$(4.1) \quad \sup \left\{ h^{\otimes n_0 a_0 + j}(l_j, l_j)(w) \mid w \in X(\mathbb{C}) \right\} \leq e^A \sup \{h(l, l)(w) \mid w \in Y(\mathbb{C})\}^{n_0 a_0 + j}$$

for $j = 0, \dots, n_0 - 1$. We choose i with $|\phi(x) - \varphi_i(x)| \leq \epsilon n_0 / 2$ for all $x \in X$, and we set $h_i = \exp(-\varphi_i) h_{FS}$. As h_i is smooth and semipositive, for the given $\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1), h_i, Y, l^{\otimes n_0}$ (as an element of $H^0(Y, \mathcal{O}_{\mathbb{P}^N}(1)|_Y)$) and $n_0 \epsilon / 4$, there is a positive integer a_1 such that the assertion of the theorem holds for all $a \geq a_1$. We put

$$n_1 := n_0 \max \left\{ a_1 + a_0 + 1, \frac{4A}{n_0 \epsilon} - 3a_0 + 1 \right\}.$$

Let n be an integer with $n \geq n_1$. If we set $n = n_0(a + a_0) + j$ ($0 \leq j \leq n_0 - 1$), then

$$a \geq a_1 \quad \text{and} \quad a \geq \frac{4A}{n_0 \epsilon} - 4a_0,$$

so that we can find $l'' \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(a))$ with $l''|_Y = l^{\otimes n_0 a}$ and

$$\begin{aligned} \sup \{h_i^{\otimes a}(l'', l'')(w) \mid w \in \mathbb{P}^N(\mathbb{C})\} \\ \leq e^{a(n_0 \epsilon/4)} \sup \{h_i(l^{\otimes n_0}, l^{\otimes n_0})(w) \mid w \in Y(\mathbb{C})\}^a, \end{aligned}$$

which implies

$$\begin{aligned} (4.2) \quad \sup \{h^{\otimes n_0 a}(l'', l'')(w) \mid w \in X(\mathbb{C})\} \\ \leq e^{(3/4)n_0 a \epsilon} \sup \{h(l, l)(w) \mid w \in Y(\mathbb{C})\}^{n_0 a} \end{aligned}$$

because of (3.1). Here we set $l' = l'' \otimes l_j$. Then, $l'|_Y = l^{\otimes n}$ and, using (4.1) and (4.2), we have

$$\begin{aligned} \sup \{h^{\otimes n}(l', l')(w) \mid w \in X(\mathbb{C})\} \\ \leq \sup \{h^{\otimes n_0 a}(l'', l'')(w) \mid w \in X(\mathbb{C})\} \sup \{h^{\otimes n_0 a_0 + j}(l_j, l_j)(w) \mid w \in X(\mathbb{C})\} \\ \leq e^{(3/4)n_0 a \epsilon + A} \sup \{h(l, l)(w) \mid w \in Y(\mathbb{C})\}^n, \end{aligned}$$

which implies the assertion because $(3/4)n_0 a \epsilon + A \leq \epsilon n$. \square

5. ARITHMETIC APPLICATION

As an application of Theorem 0.2, we have the following generalization of arithmetic Nakai-Moishezon's criterion (c.f. [5, Corollary 4.8]).

Corollary 5.1. *Let \mathcal{X} be a projective and flat integral scheme over \mathbb{Z} . Let \mathcal{L} be an invertible sheaf on \mathcal{X} such that \mathcal{L} is nef on every fiber of $\mathcal{X} \rightarrow \mathbb{Z}$. Let h be an F_∞ -invariant semipositive continuous hermitian metric of \mathcal{L} , where F_∞ is the complex conjugation map $\mathcal{X}(\mathbb{C}) \rightarrow \mathcal{X}(\mathbb{C})$. If $\widehat{\deg}(\widehat{c}_1((\mathcal{L}, h)|_{\mathcal{Y}})^{\dim \mathcal{Y}}) > 0$ for all horizontal integral subschemes \mathcal{Y} of \mathcal{X} , then, for an F_∞ -invariant continuous hermitian invertible sheaf (\mathcal{M}, k) on \mathcal{X} , $H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{M})$ has a basis consisting of strictly small sections for a sufficiently large integer n .*

Proof. Let X be the generic fiber of $\mathcal{X} \rightarrow \text{Spec}(\mathbb{Z})$ and let Y be a subvariety of X . Let \mathcal{Y} be the Zariski closure of Y in \mathcal{X} . As

$$\widehat{\deg}(\widehat{c}_1((\mathcal{L}, h)|_{\mathcal{Y}})^{\dim \mathcal{Y}}) > 0,$$

$(\mathcal{L}, h)|_{\mathcal{Y}}$ is big by [4, Theorem 6.6.1], so that $H^0(\mathcal{Y}, \mathcal{L}^{\otimes n_0}|_{\mathcal{Y}}) \setminus \{0\}$ has a strictly small section for a sufficiently large integer n_0 . Moreover, if we set $L = \mathcal{L}|_X$, then $L|_Y$ is big, and hence $\deg(L^{\dim Y} \cdot Y) > 0$ because L is nef. Therefore, L is ample by Nakai-Moishezon's criterion for ampleness. In particular, by Theorem 0.2, h is semiample metrized. Thus the assertion follows from the arguments in [5, Theorem 4.2]. \square

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